Sequences and Series 2

1. It is given that the 14th term of an arithmetic series is three times the 8th term. The 3rd term is less than the 6th term by 5. Find the common difference and the 1st term.

$T(14) = a + 13d, T(8) = a + 7d$
$T(n) = 3T(8) \Rightarrow a + (n-1)d = 3[a + 7d] \Rightarrow a = d$

$T(6) - T(3) = 5 \Rightarrow (a + 5d) - (a + 2d) = 5 \Rightarrow 3d = 5 \Rightarrow d = \frac{5}{3}.$

$$a = 10$$

2. Find the sum of the series $20^2 - 19^2 + 18^2 - 17^2 + 16^2 - 15^2 + \cdots + 2^2 - 1^2$.

$20^2 - 19^2 + 18^2 - 17^2 + 16^2 - 15^2 + \cdots + 2^2 - 1^2$
$= (20^2 - 19^2) + (18^2 - 17^2) + (16^2 - 15^2) + \cdots + (2^2 - 1^2)$
$= (20 + 19)(20 - 19) + (18 + 17)(18 - 17) + (16 + 15)(16 - 15) + \cdots + (2 + 1)(2 - 1)$
$= (20 + 19)(1) + (18 + 17)(1) + (16 + 15)(1) + \cdots + (2 + 1)(1)$
$= 20 + 19 + 18 + 17 + 16 + 15 + \cdots + 2 + 1$

$= \frac{20(20+1)}{2} = \mathbf{210}$

3. The kth term of a sequence is r_k where $r_k = 2^k + 6k - 1$. Show that the sum of r_k is the sum of a geometric series and an arithmetic series. Hence, determine the sum of the first 15 terms of the sequence.

$S(n) = \sum_{k=1}^{n} r_k = \sum_{k=1}^{n}(2^k + 6k - 1) = \sum_{k=1}^{n} 2^k + \sum_{k=1}^{n}(6k - 1)$
$\quad = (2 + 2^2 + 2^3 + \cdots + 2^n) + \big(5 + 11 + 17 + \cdots + (6n - 1)\big)$

$\quad = 2\left(\frac{2^n - 1}{2 - 1}\right) + \frac{n[5 + (6n - 1)]}{2} = (2^{n+1} - 2) + n[3n + 2]$

$S(15) = (2^{15+1} - 2) + 15[3(15) + 2] = \mathbf{66239}$

4. In an arithmetic series, the sum of the first n even number terms exceed the sum of first n odd number terms by $3n$. Find the common difference of this series.

$(T_2 + T_4 + T_6 + \cdots + T_{2n}) - (T_1 + T_3 + T_5 + \cdots + T_{2n-1})$
$= (T_2 - T_1) + (T_4 - T_3) + (T_6 - T_5) + \cdots + (T_{2n} - T_{2n-1})$
$= nd$
Therefore $nd = 3n, \quad d = \mathbf{3}.$

5. The population of a colony of bees increase in such a way that if it is N at the beginning of the week, then at the end of the week it is $a + bN$, where a and b are constants and 0 < b < 1. Starting from the beginning of the week when the population is N, write down an expression for the population at the end of 2 weeks.

Show that at the end of c consecutive weeks the population is $a\left(\frac{1-b^c}{1-b}\right) + b^c N$.

When a = 2000 and b = 0.2, it is known that the population takes about 4 weeks to increase from N and 2N. Estimate a value for N from this information.

The population at the end of 2 weeks = $a + b(a + bN) = a + ab + b^2N$

The population at the end of 3 weeks = $a + b(a + ab + b^2N) = a + ab + ab^2 + b^3N$

....

The population at the end of c consecutive weeks

$= (a + ab + ab^2 + \cdots + ab^{c-1}) + b^c N$

$= a\left(\frac{1-b^c}{1-b}\right) + b^c N,$ G.P. first term is a and common ratio is b.

When a = 2000 and b = 0.2, c = 4

The population at the end of c consecutive weeks = $2000\left(\frac{1-0.2^4}{1-0.2}\right) + 0.2^4 N = 2N$

$2500(1 - 0.2^4) + 0.2^4 N = 2N$

$2500(1 - 0.2^4) = N(2 - 0.2^4)$

$N = \frac{2500(1-0.2^4)}{2-0.2^4} \approx 1248.9991993594876 \approx \mathbf{1249}$

6. Verify that $4r^3 + r = \left(r + \frac{1}{2}\right)^4 - \left(r - \frac{1}{2}\right)^4$. Hence, find $\sum_{r=1}^{n}(4r^3 + r)$.

Deduce that $\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n + 1)^2$.

$$\left(r + \frac{1}{2}\right)^4 - \left(r - \frac{1}{2}\right)^4 = \left[\left(r + \frac{1}{2}\right)^2 - \left(r - \frac{1}{2}\right)^2\right]\left[\left(r + \frac{1}{2}\right)^2 + \left(r - \frac{1}{2}\right)^2\right] = [2r]\left[2r^2 + \frac{1}{2}\right] = 4r^3 + r$$

$$\sum_{r=1}^{n}(4r^3 + r) = \sum_{r=1}^{n}\left[\left(r + \frac{1}{2}\right)^4 - \left(r - \frac{1}{2}\right)^4\right]$$

$$= \left[\left(n + \frac{1}{2}\right)^4 - \left(n - \frac{1}{2}\right)^4\right] + \left[\left(n - \frac{1}{2}\right)^4 - \left(n - \frac{3}{2}\right)^4\right] + \left[\left(n - \frac{3}{2}\right)^4 - \left(n - \frac{5}{2}\right)^4\right] + \cdots \left[\frac{3^4}{2} - \frac{1^4}{2}\right]$$

$$= \left(n + \frac{1}{2}\right)^4 - \frac{1^4}{2}$$

$$\sum_{r=1}^{n}(4r^3 + r) = \sum_{r=1}^{n} 4r^3 + \sum_{r=1}^{n} r = \left(n + \frac{1}{2}\right)^4 - \frac{1^4}{2}$$

$4\sum_{r=1}^{n} r^3 + \frac{n(n+1)}{2} = \left(n+\frac{1}{2}\right)^4 - \frac{1^4}{2}$

$4\sum_{r=1}^{n} r^3 = \left(n+\frac{1}{2}\right)^4 - \frac{1^4}{2} - \frac{n(n+1)}{2} = n^2(n+1)^2$

$\therefore \sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$

7. If $f(r) = \cos 2r\theta$, simplify $f(r) - f(r-1)$. Use your result to find the sum of the first n terms of the series $\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots$

$f(r) - f(r-1) = \cos 2r\theta - \cos 2(r-1)\theta = -\frac{1}{2} sin \frac{2r\theta + 2(r-1)\theta}{2} sin \frac{2r\theta - 2(r-1)\theta}{2}$

$= -\frac{1}{2}\sin(2r-1)\theta \sin\theta$

$\sum_{r=2}^{n}[f(r) - f(r-1)] = -\frac{1}{2}\sin\theta \sum_{r=2}^{n} \sin(2r-1)\theta$

$f(n) - f(1) = -\frac{1}{2}\sin\theta\,(\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta)$

$\cos 2n\theta - \cos 2\theta = -\frac{1}{2}\sin\theta\,[\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta]$

$\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta = 2\left(\frac{\cos 2\theta - \cos 2n\theta}{\sin\theta}\right)$

Sum of the first n terms of the series $\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots$

$= \sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta + \sin(2n+1)\theta = \mathbf{2}\left[\frac{\mathbf{\cos 2\theta - \cos 2(n+1)\theta}}{\mathbf{\sin\theta}}\right]$

8. Show that the following infinite series is geometric and find its sum:

$2^2(1-2x)^2 + 2^3(1-2x)^3 + 2^4(1-2x)^4 + \cdots + 2^r(1-2x)^r + \cdots$

and state the values of x for which the result is valid.

$\frac{T(r+1)}{T(r)} = \frac{2^{r+1}(1-2x)^{r+1}}{2^r(1-2x)^r} = 2(1-2x)$ is independent of r and therefore the given series is a G.P.

Common ratio is $2(1-2x)$. $S(\infty) = \frac{a}{1-r}$

$2^2(1-2x)^2 + 2^3(1-2x)^3 + 2^4(1-2x)^4 + \cdots + 2^r(1-2x)^r + \cdots = \frac{2^2(1-2x)^2}{1-2(1-2x)} = \frac{\mathbf{4(1-2x)^2}}{\mathbf{4x-1}}$

Validity for x : $|2(1-2x)| < 1 \Leftrightarrow |2x-1| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < 2x - 1 < \frac{1}{2} \Leftrightarrow \frac{\mathbf{1}}{\mathbf{4}} < x < \frac{\mathbf{3}}{\mathbf{4}}$

9. An education fund was set up by a school with an initial sum of \$20000 to award \$1800 annually to the best student. The fund is kept in a bank paying a simple interest of 6% per year. If the first award is given out exactly a year after the fund was set up in the bank, fund the number of awards that can be given out continuously.

After first year, amount left in the bank = $20000(1.06) - 1800$

After second year, amount left in the bank = $[20000(1.06) - 1800](1.06) - 1800$

$= 20000(1.06)^2 - 1800(1 + 1.06)$

After third year, amount left in the bank = $[20000(1.06)^2 - 1800(1 + 1.06)](1.06) - 1800$

$= 20000(1.06)^3 - 1800(1 + 1.06 + 1.06^2)$

After n years, amount left = $20000(1.06)^n - 1800(1 + 1.06 + 1.06^2 + \cdots + 1.06^{n-1})$

$$= 20000(1.06)^n - 1800\frac{1.06^n - 1}{1,06 - 1} = 20000(1.06)^n - 30000(1.06^n - 1)$$

$$= 30000 - 10000(1.06)^n$$

After n years, amount left ≈ 0

$30000 - 10000(1.06)^n \approx 0$

$(1.06)^n \approx 3$

$n\,log\ 1.06 \approx \log 3$

$n \approx \frac{\log 3}{log\ 1.06} \approx 18.8541766791073$

The number of awards that can be given out is 18 years and there is a small sum in the bank that cannot support the next award.

10. The first three terms of an arithmetic series are $-\frac{9}{8}, -\frac{5}{8}, -\frac{1}{8}$.

Show that the sum, S_n of the first n terms of this series is $\frac{1}{8}n(2n - 11)$ for $n = 1,2,3, \ldots$

Find **(i)** the value of n when S_n is the smallest,

(ii) set of values of n where $0 < S_n < 10$.

Common difference = $-\frac{5}{8} - \left(-\frac{9}{8}\right) = \frac{1}{2}$, First term = $-\frac{9}{8}$.

Hence $S_n = \frac{1}{2}n\left[2\left(-\frac{9}{8}\right) + (n-1)\frac{1}{2}\right] = \frac{1}{8}n[-9 + 2(n-1)] = \boldsymbol{\frac{1}{8}n(2n - 11)}$

(i) $S_n = \frac{1}{8}n(2n - 11) = \frac{1}{4}\left[n^2 - \frac{11}{2}n\right] = \frac{1}{4}\left[n^2 - 2\left(\frac{11}{4}\right)n + \left(\frac{11}{4}\right)^2 - \left(\frac{11}{4}\right)^2\right] = \frac{1}{4}\left[\left(n - \frac{11}{4}\right)^2 - \left(\frac{11}{4}\right)^2\right]$

Hence S_n is smallest when $n = \frac{11}{4}$, and the smallest S_n is $-\frac{1}{4}\left(\frac{11}{4}\right)^2 = -\boldsymbol{\frac{121}{256}}$.

(ii) For $S_n > 0$, $\frac{1}{8}n(2n-11) > 0 \Longrightarrow n < 0 \ or \ n > \frac{11}{2} \Longrightarrow n > \frac{11}{2} = 5.5$, since $n > 0$.

For $S_n < 10 \Longrightarrow \frac{1}{8}n(2n-11) < 10 \Longrightarrow 2n^2 - 11n - 80 < 0$

$$\Longrightarrow \frac{11-\sqrt{761}}{4} < n < \frac{\sqrt{761}+11}{4} \Longrightarrow 0 < n < 9.65$$

So for $0 < S_n < 10$, combining the above result, we have $5.5 < n < 9.65$

Since n is an integer , we have $n = \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}$.

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